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Free Fermions and Extended Conformal Algebras

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ABSTRACT

A class of algebras is constructed using free fermions and the invariant anti-symmetric tensors associated with irreducible holonomy groups.

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Given a set of free fermions $\{\lambda^a; a = 1, \dots, n\}$ in two dimensions, one can construct a set of conserved currents $\{J^{(p)}; p = 1, \dots, n\}$ where $J^{(p)}$ is the normal-ordered product of p fermion fields (with uncontracted indices). Thus each current is an antisymmetric tensor in an associated n -dimensional vector space V ($\dim V = n$). The set of currents which are quadratic in the fermion fields ($p = 2$) form an $SO(n)$ Kac-Moody current. As such, it is natural to ask whether there are other current algebras constructed by taking higher order polynomials of these currents. Indeed it was observed [1] that one could generate the $N = 1$ superconformal algebra by constructing the supercharge from the three fermion current contracted with the structure constant of a Lie algebra.

In this note we will show that certain subalgebras of this larger algebra may be extracted using antisymmetric tensors of V that are invariant under a subgroup G of $O(n)$ acting on V with some representation R . Invariant forms can exist in every irreducible representation of a compact Lie group. To be more precise, we shall use the invariant forms which occur in the fundamental representation of irreducible holonomy groups of non-symmetric Riemannian manifolds. We will also use the invariant forms of the adjoint representation of these groups. Typically we find algebras with additional spin $3/2$ and spin 2 generators. For $SO(n)$ and $SU(n/2)$ higher spin currents arise for large enough n . In these cases, one does not generate a closed algebra using only the invariant tensors (and T) although it might be possible to achieve closure by including a finite number of additional currents.

We recall that the possible irreducible holonomy groups of n -dimensional non-symmetric Riemannian manifolds are given by Berger's list [2]: $SO(n)$, $U(n/2)$, $SU(n/2)$, $Sp(n/4)$, $Sp(n/4) \cdot Sp(1)$ and two exceptional cases, G_2 ($n = 7$) and $Spin(7)$ ($n=8$). The corresponding invariant forms are: the ϵ -tensor ($SO(n)$); the Kahler 2-form ($U(n/2)$), the holomorphic ϵ -tensor ($SU(n/2)$), the three Kahler 2-forms ($Sp(n/4)$), and for $Sp(n/4) \cdot Sp(1)$, a four-form which is locally the sum of the wedge product of each of the three local Kahler forms with itself. In G_2 , there is a 3-form and its 4-form dual, and in $Spin(7)$ a self-dual 4-form. It has been noted

[3] that (two-dimensional) supersymmetric sigma models with these manifolds as target spaces have additional symmetries associated with these form; and that the corresponding currents belong, at least classically, to extended superconformal algebras which close in a non-linear fashion, i.e. they are of the W-symmetry type. The $SU(3)$, G_2 and $Spin(7)$ cases are relevant in string theory compactifications to 4, 3 and 2 dimensions respectively (see for example [5]).

Let us now turn to a free fermion theory with n fermions $\{\lambda^a\}$. In the general case, we can introduce the currents

$$J_{a_1 \dots a_p} =: \lambda_{a_1} \dots \lambda_{a_p} : \quad (1)$$

The OPE of two such currents (with $q \leq p$) is

$$J_{a_1 \dots a_p}(z) J_{b_1 \dots b_q}(w) = \sum_{m=1}^q \frac{(-1)^{pm + \frac{m(m-1)}{2}}}{(z-w)^m} \delta_{[a_1 \dots a_m}^{[b_1 \dots b_m} \lambda_{a_{m+1} \dots a_p]}(z) \lambda^{b_{m+1} \dots b_q]}(w) \quad (2)$$

The antisymmetry symbols in the above equation are with weight one. Clearly, this O.P.E algebra closes in the sense that all the terms on the right-hand-side can be arranged into products of J 's and their derivatives.

To extract the sub-algebras of interest we shall contract some of the above J 's with the invariant tensors of the holonomy groups G which we have listed above. Let

$$g = g_{a_1 \dots a_k} e^{a_1} \wedge \dots \wedge e^{a_k}$$

be a (constant) k -form of the vector space V and $\{e^a\}$, $a = 1, \dots, \dim V$ a basis of V . The currents that we will study are

$$X = \frac{1}{k!} g_{a_1 \dots a_k} : \lambda_{a_1} \dots \lambda_{a_k} : \quad (3)$$

All these currents are primary with conformal spin $\frac{k}{2}$ with respect to the energy

momentum

$$T = \frac{1}{2} \lambda^a \partial \lambda_a$$

of the free fermions, i.e. the O.P.E of T and X is

$$T(z)X(w) = \frac{\partial X}{z-w} + \frac{k}{2} \frac{X}{(z-w)^2}. \quad (4)$$

1. $G = SO(4)$

We define

$$X = \frac{1}{4!} \epsilon_{abcd} : \lambda^a \lambda^b \lambda^c \lambda^d : \quad (5)$$

we then find

$$X(z)X(w) = \frac{1}{(z-w)^4} + \frac{2T}{(z-w)^2} + \frac{\partial T}{(z-w)} \quad (6)$$

The O.P.E of T with X closes as in equation (4). Thus, we have an algebra with two spin 2 currents. It might be thought that this algebra should factorise, but it does not as one may verify by considering all possible linear combinations of X and T .

If one goes to higher n , one finds that the OPE algebra does not close on T and the spin $n/2$ current

$$X = \frac{1}{n!} \epsilon_{abcde\dots} : \lambda^a \lambda^b \lambda^c \lambda^d \lambda^e \dots : \quad (7)$$

Indeed, if one considers the OPE of X with itself, the two most singular terms, are

$$\frac{(-1)^{\frac{n(3n-1)}{2}}}{(z-w)^n} - \frac{(-1)^{\frac{n(3n+1)}{2}}}{(z-w)^{n-2}} \cdot \frac{1}{2} \cdot \lambda^{ab}(z) \lambda_{ab}(w). \quad (8)$$

Up to a sign we can identify the coefficient of the latter term as $2T^2$. However, were the algebra to close on X and T alone then the coefficient of the above T^2

term is fixed [4] , by conformal invariance, to be $2 \cdot \frac{5s+1}{22+5c}$ in magnitude, where, for us, $c = s = \frac{n}{2}$. Clearly, the above O.P.E. can not close on X and T alone. Explicit calculation of the full O.P.E. for $SO(5)$ bears out this conclusion. This does not mean that the O.P.E. does not close if one includes more currents or a different choice for energy-momentum tensor.

2. $G = SU(3)$

In $SU(3)$ we have $n = 6$ λ 's and replace them by 3 complex ones, i.e. $\lambda^a \rightarrow \{\lambda^\alpha, \bar{\lambda}^\alpha\}$, $\alpha = 1, 2, 3$. There is complex spin 3/2 current

$$G = \frac{1}{3!} \epsilon_{\alpha\beta\gamma} : \lambda^\alpha \lambda^\beta \lambda^\gamma : \quad (9)$$

and its conjugate

$$\bar{G} = \frac{1}{3!} \epsilon_{\alpha\beta\gamma} : \bar{\lambda}^\alpha \bar{\lambda}^\beta \bar{\lambda}^\gamma : \quad (10)$$

One finds

$$G(z)\bar{G}(w) = \frac{1}{(z-w)^3} - \frac{J}{(z-w)^2} + \frac{\{-\frac{1}{2}\partial J + T'\}}{(z-w)} \quad (11)$$

where

$$J = : \lambda^\alpha \bar{\lambda}^\alpha : \quad T' = T + \frac{1}{2} : \lambda^\beta \bar{\lambda}_\beta \lambda^\gamma \bar{\lambda}_\gamma : \quad (12)$$

While J is a $U(1)$ current, one can show by re-normal ordering the λ^4 term that $T' = \frac{1}{2} : J^2 :$ and so is the Sugawara energy-momentum tensor associated with the $U(1)$ current. Evaluating the other O.P.E's and, making a suitable rescaling, one finds that T', G, \bar{G} and J define an $N = 2$ superconformal algebra with central charge $c = 1$. The field $T - \frac{T'}{3}$ commutes with this algebra. This algebra occurs as a subalgebra of the extended $N = 2$ superconformal algebra which arises in the context of Calabi-Yau compactifications [6].

For the $SU(n)$ case, we have the spin $\frac{n}{2}$ current

$$X = \frac{1}{n!} \epsilon_{\alpha\beta\gamma\delta\dots} : \lambda^\alpha \lambda^\beta \lambda^\gamma \lambda^\delta \dots : \quad (13)$$

and its complex conjugate

$$\bar{X} = \frac{1}{n!} \epsilon_{\alpha\beta\gamma\delta\dots} : \bar{\lambda}^\alpha \bar{\lambda}^\beta \bar{\lambda}^\gamma \bar{\lambda}^\delta \dots : \quad (14)$$

The OPE of X with its complex conjugate does not close on T for the reasons given earlier, however one may hope that it will close using Sugawara-like currents. This possibility remains under study.

3. $Sp(k) \cdot Sp(1)$

This is the holonomy group of quaternionic Kahler manifolds in which case there are three (locally defined) complex structures I_r that obey the algebra of imaginary unit quaternions ($I_r I_s = -\delta_{rs} + \sum_t \epsilon_{rst} I_t$) and three associated Kahler forms ω_r . From these one can define an $Sp(k) \cdot Sp(1)$ invariant 4-form

$$\Omega = \sum_{r=1}^3 \omega_r \wedge \omega_r \quad (15)$$

The associated spin 2 current current is

$$X = \frac{1}{4!} \Omega_{abcd} : \lambda^a \lambda^b \lambda^c \lambda^d : \quad (16)$$

One finds that the OPE algebra is

$$\begin{aligned} X(z)X(w) = & \frac{1}{4!} \frac{n(n+2)}{(z-w)^4} + \frac{(n+2)}{3} \frac{T(w)}{(z-w)^2} + \frac{(n+2)}{6} \frac{\partial T(w)}{z-w} - \frac{n-4}{6} \frac{\partial X(w)}{z-w} \\ & - \frac{n-4}{3} \frac{X(w)}{(z-w)^2}, \end{aligned} \quad (17)$$

where $n = 4k$.

4. G_2 and $\text{Spin}(7)$

The final examples are provided by the exceptional holonomy groups. In G_2 there is a 3-form (e^a a basis in \mathbb{R}^7)

$$\phi = f_{abc}e^a \wedge e^b \wedge e^c \quad (18)$$

and a 4-form which is the dual of ϕ

$$*\phi = f_{abcd}e^a \wedge e^b \wedge e^c \wedge e^d \quad (19)$$

Using these we can form the currents

$$G = \frac{1}{3!}f_{abc} : \lambda^a \lambda^b \lambda^c : \quad (20)$$

and

$$X = \frac{1}{4!}f_{abcd} : \lambda^a \lambda^b \lambda^c \lambda^d : \quad (21)$$

The OPE algebra is

$$\begin{aligned} G(z)G(w) &= \frac{7}{(z-w)^3} + \frac{6T}{(z-w)} - \frac{6X}{(z-w)} \\ G(z)X(w) &= -\frac{6G(w)}{(z-w)^2} - \frac{2\partial G}{z-w} \\ X(z)X(w) &= \frac{7}{(z-w)^4} + \frac{8T}{(z-w)^2} + \frac{4\partial T}{z-w} - \frac{6X}{(z-w)^2} - \frac{3\partial X}{z-w}. \end{aligned} \quad (22)$$

In the $\text{Spin}(7)$ case ($n = 8$), one has an invariant 4-form which is closely related to the forms of G_2 ;

$$\phi = g_{abcd} : \lambda^a \lambda^b \lambda^c \lambda^d : \quad (23)$$

with $g_{0abc} = f_{abc}$ (for $a, b, c = 1, \dots, 7$) and $g_{abcd} = f_{abcd}$ (for $a, b, c, d = 1, \dots, 7$).

The associated spin 2 current is

$$X = \frac{1}{4!} g_{abcd} : \lambda^a \lambda^b \lambda^c \lambda^d : \quad (24)$$

The O.P.E is

$$X(z)X(w) = \frac{14}{(z-w)^4} + \frac{14T}{(z-w)^2} + \frac{7\partial T}{z-w} - \frac{12X}{(z-w)^2} - \frac{6\partial X}{z-w} \quad (25)$$

Note Added

This paper was based on work carried out a number of years ago, we were motivated to publish it by the recent appearance of the paper "Superstrings and Manifolds of Exceptional Holonomy ". by S. Shatashvili and C. Vafa in which the G_2 and $\text{spin}(7)$ examples are discussed in a free superfield realisation.

References

- 1 P. di Vecchia, V. Knizhnik, J. Peterson, P. Rossi; Nucl Phys B253 (1985) 157.
- 2 M. Berger, Bull. Soc. Math. France 83 (1953) 279
- 3 P. S. Howe and G.Papadopoulos, Phys.lett. 263B (1991) 230.
P. S. Howe and G.Papadopoulos, Phys.lett .267B (1991) 362.
P. S. Howe and G.Papadopoulos, Comm Math Phys 151 (1993) 467.
- 4 A.B. Zamolodchikov, Theor. Math. Phys 65 (1989) 1205.
- 5 P. Candelas, G. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B258 (1985) 46.
C. M. Hull, " Superstring Compactifications with Torsion and space-time Supersymmetry", Published in Turin conference on "Superunification" (1985) 347.
- 6 S. Odake, Mod. Phys. Lett A4 (1989) 557.